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Bundles in Classical Gauge Field Theory

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Outline

Definition

- Let $\mathcal{M}$ be a pseudo-Riemannian manifold with a metric g. A classical field ϕ of rank (p, q) is a differentiable tensor field living on $\mathcal{M}$ i.e.,

As a differentiable tensor field

- $\phi : \mathcal{M} \to \left(\prod_{i=1}^{p} V^* \times \prod_{j=1}^{q} V \to \mathbb{R} \right)$
- $\phi \in C(\mathcal{M})$

where V is a vector space with $\mathbb{R}$ as the base field.

- This is the starting point for defining classical fields. Additionally, they obey some physical properties discussed below.

Physical properties

1. Stationary-action principle
2. Local Lorentz invariance
3. Gauge invariance

Stationary-action Principle

- Let the function space of ϕ, i.e. $\left[\mathcal{M} \to \left(\prod_{i=1}^{p} V^* \times \prod_{j=1}^{q} V \to \mathbb{R}\right)\right] \cap C\left(\mathcal{M}\right)$, be denoted as $\mathcal{F}$.

Definition (Lagrangian)

The Lagrangian [density] $\mathcal{L}$ of a classical field ϕ is a differentiable map $\mathcal{L} : \mathcal{F} \times T^*\mathcal{M} \times \mathcal{M} \to \mathbb{R}$.

Here, $T^*\mathcal{M}$ denotes the cotangent bundle of $\mathcal{M}$.

- However, we have not yet motivated bundles. Therefore, for now, we will think of $T^*\mathcal{M}$ as being set-theoretically isomorphic to the set of covariant derivatives of ϕ along every continuous curve γ in $\mathcal{M}$,

$$T^*\mathcal{M} \cong_{\text{set}} \{\nabla_\gamma \phi \mid \gamma : [0,1] \to \mathcal{M} \text{ is continuous}\}$$

- By continuous curves, we refer to the topological notion of the continuity of maps from the topological space $\left([0,1], \mathcal{O}_{\mathbb{R}}\big|_{[0,1]}\right)$ to $(\mathcal{M}, \mathcal{O}_{\mathcal{M}})$. Here, $\mathcal{O}_{\mathbb{R}}\big|_{[0,1]}$ is the subspace topology induced on the unit interval by the Euclidean topology on $\mathbb{R}$ and $\mathcal{O}_{\mathcal{M}}$ is the manifold topology on $\mathcal{M}$.

Definition (Action)

The action for a tensor field ϕ in a compact neighbourhood $U \subset \mathcal{M}$ is the linear functional,

$$S[\phi] := \int_{x \in U} \varepsilon \mathcal{L}\left(\phi(x), T_x^* \mathcal{M}, x\right)$$

where ε is the Riemannian volume form which in local coordinates can be written as,

$$\varepsilon := \sqrt{|\det(g)|} \bigwedge_{\mu} \mathrm{d}x^{\mu}$$

In local coordinates, using index notation, the action can be covariantly written in terms of components as,

$$S[\phi(x^{\alpha})] = \int_U \varepsilon \mathcal{L}\left(\phi^{\rho_1 \cdots \rho_p}{}_{\lambda_1 \ldots \lambda_q}, \nabla_{\mu} \phi^{\rho_1 \cdots \rho_p}{}_{\lambda_1 \ldots \lambda_q}, x^{\alpha}\right)$$

where $\phi^{\rho_1 \cdots \rho_p}{}_{\lambda_1 \ldots \lambda_q} = \bigcirc_{i=1}^{p} \mathrm{d}x^{\rho_i} \circ \bigcirc_{j=1}^{q} \partial_{\lambda_j}(\phi)$.

Postulate (Stationary-principle action)

For on-shell trajectories $\phi \in \mathcal{F}$, we have the following for all compact neighbourhoods $U \subset \mathcal{M}$,

$$\delta S\left[\phi\right] = 0$$

i.e.,

$$\delta \int_U \varepsilon \mathcal{L}\left(\phi^{\rho_1 \cdots \rho_p}{}_{\lambda_1 \ldots \lambda_q}, \nabla_\mu \phi^{\rho_1 \cdots \rho_p}{}_{\lambda_1 \ldots \lambda_q}, x^\alpha\right) = 0$$

Theorem (Euler-Lagrange equations)

A classical field ϕ is on-shell i.e. obeys the principle of stationary action if and only if it satisfies the Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \phi^{\rho_1 \cdots \rho_p}{}_{\lambda_1 \ldots \lambda_q}} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial \left(\nabla_\mu \phi^{\rho_1 \cdots \rho_p}{}_{\lambda_1 \ldots \lambda_q}\right)} = 0$$

with summation over dummy indices implied (Einstein summation convention).

Proof.

$$\delta S = 0$$

$$\delta \int_U \varepsilon \mathcal{L} = 0$$

$$\int_U \varepsilon \left[\delta \phi^{\rho_1 \cdots \rho_p}{}_{\lambda_1 \ldots \lambda_q} \frac{\partial \mathcal{L}}{\partial \phi^{\rho_1 \cdots \rho_p}{}_{\lambda_1 \ldots \lambda_q}} + \delta \left(\nabla_\mu \phi^{\rho_1 \cdots \rho_p}{}_{\lambda_1 \ldots \lambda_q} \right) \frac{\partial \mathcal{L}}{\partial \left(\nabla_\mu \phi^{\rho_1 \cdots \rho_p}{}_{\lambda_1 \ldots \lambda_q} \right)} \right] = 0$$

$$\int_U \varepsilon \left[\delta \phi^{\rho_1 \cdots \rho_p}{}_{\lambda_1 \ldots \lambda_q} \frac{\partial \mathcal{L}}{\partial \phi^{\rho_1 \cdots \rho_p}{}_{\lambda_1 \ldots \lambda_q}} + \nabla_\mu \left(\delta \phi^{\rho_1 \cdots \rho_p}{}_{\lambda_1 \ldots \lambda_q} \right) \frac{\partial \mathcal{L}}{\partial \left(\nabla_\mu \phi^{\rho_1 \cdots \rho_p}{}_{\lambda_1 \ldots \lambda_q} \right)} \right] = 0$$

Proof (continued).

$$
\int_U \varepsilon \delta\phi^{\rho_1\cdots\rho_p}{}_{\lambda_1\ldots\lambda_q} \frac{\partial\mathcal{L}}{\partial\phi^{\rho_1\cdots\rho_p}{}_{\lambda_1\ldots\lambda_q}}
+ \cancel{\frac{\partial\mathcal{L}}{\partial\left(\nabla_\mu\phi^{\rho_1\cdots\rho_p}{}_{\lambda_1\ldots\lambda_q}\right)} \int_U \varepsilon\nabla_\mu\left(\delta\phi^{\rho_1\cdots\rho_p}{}_{\lambda_1\ldots\lambda_q}\right)}
- \int_U \varepsilon\left[\nabla_\mu \frac{\partial\mathcal{L}}{\partial\left(\nabla_\mu\phi^{\rho_1\cdots\rho_p}{}_{\lambda_1\ldots\lambda_q}\right)} \int \varepsilon\nabla_\mu\left(\delta\phi^{\rho_1\cdots\rho_p}{}_{\lambda_1\ldots\lambda_q}\right)\right] = 0
$$

$$
\int_U \varepsilon \delta\phi^{\rho_1\cdots\rho_p}{}_{\lambda_1\ldots\lambda_q} \frac{\partial\mathcal{L}}{\partial\phi^{\rho_1\cdots\rho_p}{}_{\lambda_1\ldots\lambda_q}}
- \int_U \varepsilon\left[\delta\phi^{\rho_1\cdots\rho_p}{}_{\lambda_1\ldots\lambda_q} \nabla_\mu \frac{\partial\mathcal{L}}{\partial\left(\nabla_\mu\phi^{\rho_1\cdots\rho_p}{}_{\lambda_1\ldots\lambda_q}\right)}\right] = 0
$$

Proof (continued).

$$\int_U \varepsilon \delta \phi^{\rho_1 \cdots \rho_p}{}_{\lambda_1 \ldots \lambda_q} \left[\frac{\partial \mathcal{L}}{\partial \phi^{\rho_1 \cdots \rho_p}{}_{\lambda_1 \ldots \lambda_q}} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial \left(\nabla_\mu \phi^{\rho_1 \cdots \rho_p}{}_{\lambda_1 \ldots \lambda_q} \right)} \right] = 0$$

Since the above is true for all compact neighbourhoods $U \subset \mathcal{M}$, by the fundamental lemma of the calculus of variations,

$$\frac{\partial \mathcal{L}}{\partial \phi^{\rho_1 \cdots \rho_p}{}_{\lambda_1 \ldots \lambda_q}} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial \left(\nabla_\mu \phi^{\rho_1 \cdots \rho_p}{}_{\lambda_1 \ldots \lambda_q} \right)} = 0 \quad \square$$

The power of the above functional-analytic manipulations and notions is that the above statements are all logically equivalent, therefore proving 'S-A principle iff E-L equations'.

Local Lorentz Invariance

- Local Lorentz invariance is the idea that at each $p \in \mathcal{M}$, the action of the restricted Lorentz group $\mathrm{SO}^+(1,3)$ on tensorial objects living on $T_p\mathcal{M}$, leaves them invariant.
- This means that the components of a rank (p,q) tensor field T with components ${T^{\rho_1 \dots \rho_p}}_{\lambda_1 \dots \lambda_q}$ must transform covariantly with respect to the restricted Lorentz group.
  In other words, we require that for any pair of primed and unprimed coordinate systems related by some transformation $\Lambda \in \mathrm{SO}^+(1,3)$, the following principle applies:

Postulate (Local Lorentz invariance)

$$T = T'$$

This simple principle has far-reaching consequences in theoretical physics, such as severe restriction induced on the form of physical laws and equations.

Theorem (Tensor component transformation law)

Invariance holds if and only if for a tensor field T, its components transform under any $\Lambda \in \mathsf{SO}^+(1,3)$ represented by (in terms of its action on the concerned tangent space) a Jacobian with components ${\Lambda^{\mu'}}_{\mu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}}$ as,

$$T^{\rho_{1'}\cdots\rho_{p'}}{}_{\lambda_{1'}\ldots\lambda_{q'}} = \left(\prod_{i=1}^{p} \Lambda^{\rho_i'}{}_{\rho_i}\right) T^{\rho_1\cdots\rho_p}{}_{\lambda_1\ldots\lambda_q} \left(\prod_{j=1}^{q} \ldots \Lambda^{\lambda_j}{}_{\lambda_j'}\right)$$

Proof.

By local Lorentz invariance,

$$\begin{aligned} T^{\rho_{1'}\cdots\rho_{p'}}{}_{\lambda_{1'}\ldots\lambda_{q'}} &:= T\left(\mathsf{d}x^{\rho_{1'}}, \ldots, \mathsf{d}x^{\rho_{p'}}, \partial_{\lambda_{1'}}, \ldots, \partial_{\lambda_{q'}}\right) \\ &= T\left(\frac{\partial x^{\rho_{1'}}}{\partial x^{\rho_1}}\mathsf{d}x^{\rho_1}, \ldots, \frac{\partial x^{\rho_{p'}}}{\partial x^{\rho_p}}\mathsf{d}x^{\rho_p}, \frac{\partial x^{\lambda_1}}{\partial \lambda_{1'}}\partial_{\lambda_1}, \ldots, \frac{\partial x^{\lambda_q}}{\partial \lambda_{q'}}\partial_{\lambda_q}\right) \end{aligned}$$

Proof (continued).

Since a tensor is a multilinear map,

$$
\begin{aligned}
T^{\rho_{1'} \cdots \rho_{p'}}{}_{\lambda_{1'} \ldots \lambda_{q'}} &= \left(\prod_{i=1}^{p} \frac{\partial x^{\rho_{i'}}}{\partial x^{\rho_i}} \right) T\left(\mathrm{d}x^{\rho_1}, \ldots, \mathrm{d}x^{\rho_p}, \partial_{\lambda_1}, \ldots, \partial_{\lambda_q} \right) \left(\prod_{j=1}^{q} \frac{\partial x^{\lambda_j}}{\partial \lambda_{j'}} \right) \\
&= \left(\prod_{i=1}^{p} \frac{\partial x^{\rho_{i'}}}{\partial x^{\rho_i}} \right) T^{\rho_1 \cdots \rho_p}{}_{\lambda_1 \ldots \lambda_q} \left(\prod_{j=1}^{q} \frac{\partial x^{\lambda_j}}{\partial \lambda_{j'}} \right) \\
&= \left(\prod_{i=1}^{p} \Lambda^{\rho_i'}{}_{\rho_i} \right) T^{\rho_1 \cdots \rho_p}{}_{\lambda_1 \ldots \lambda_q} \left(\prod_{j=1}^{q} \ldots \Lambda^{\lambda_j}{}_{\lambda_j'} \right) \qquad \square
\end{aligned}
$$

Gauge Invariance

Observational equivalence

In classical field theory, observational equivalence is the idea that two classical fields ψ and ϕ yielding identical physical quantities give rise to identical physical predictions.

- Typically, these physical quantities are geometric objects such as the curvature form $\Omega = \mathrm{d}\phi + \phi \wedge \phi$ associated with ϕ.
- This gives rise to gauge freedom, wherein a classical field can contain physically redundant information in its representation as a differentiable tensor field.
- Therefore, given actual physical quantities in some context, such as the curvature form, there arise multiple ways to write the underlying classical field, each representation said to be a 'gauge' of the field.

Definition (Gauge of a classical field)

Formally, a gauge of a classical field ϕ can be thought of as some representative of the equivalence class $[\phi]$ defined by some equivalence relation (gauge invariance) of the form,

$$\forall \psi, \phi \in \mathcal{F} : \psi \sim \phi :\Longleftrightarrow \exists f \in G : f \cdot \psi = \phi$$

where $(G, \cdot)$ is some group (called the gauge group of the concerned field) which preserves relevant physical quantities such as curvature.

- e.g. Consider the Newtonian gravitational field ϕ, which is a real-valued scalar field on a 3-dimensional *pseudo*-Riemannian manifold $\mathcal{M}$. Its curvature form is,

$$\begin{aligned}\Omega &= \mathrm{d}\phi + \cancel{\phi \wedge \phi} \\ &= \mathrm{d}\phi\end{aligned}$$

  In local coordinates, the components of $\Omega = \mathrm{d}\phi$ are $\Omega_i = \partial_i \phi$. This is identical (up to scaling) to the dual of the gravitational force field F^*. I.e.,

$$\begin{aligned}F^* &= -m\mathrm{d}\phi \\ F_i &= -m\partial_i \phi\end{aligned}$$

- Since the force field is a physical entity, any gauge transformation of ϕ leaving its curvature form invariant, must be observationally equivalent to ϕ. An example of such a transformation is a translation disctated by the additive group of closed 1-forms ω,

$$\begin{aligned}\phi \mapsto \widetilde{\phi} &= \phi + \omega \\ \Omega \mapsto \widetilde{\Omega} &= \mathrm{d}\widetilde{\phi} \\ &= \mathrm{d}\left(\phi + \omega\right) \\ &= \mathrm{d}\phi + \cancel{\mathrm{d}\omega} \\ &= \Omega\end{aligned}$$

- Similarly, in electromagnetism, a gauge transformation of the potential 1-form A resembles translation under the additive group of 1-forms. This leaves the curvature form $F = \mathrm{d}A$ invariant,

$$\begin{aligned}A \mapsto \widetilde{A} &= A + \mathrm{d}\alpha \\ F \mapsto \widetilde{F} &= \mathrm{d}\widetilde{A} \\ &= \mathrm{d}\left(A + \mathrm{d}\alpha\right) \\ &= \mathrm{d}A + \cancel{\mathrm{d}^2\alpha} \\ &= F\end{aligned}$$

Fibres

Definition (Fibre)

The fibre $F\left(p\right)$ associated with a classical field ϕ, at a point $p \in \mathcal{M}$ is defined as,

$$F\left(p\right) := \bigcup_{\psi \in [\phi]} \left\{\left(p, \psi\left(p\right)\right)\right\}$$

Intuitively, the fibre at a point is simply the set of values of the classical field in all its gauges, at that point.

Total Space

Definition (Total space)

The total space E associated with a classical field ϕ living on a spacetime $\mathcal{M}$ is defined as,

$$E := \bigcup_{p \in \mathcal{M}} F(p)$$

Remark

$$\begin{aligned} E &= \bigcup_{p \in \mathcal{M}} F(p) \\ &= \bigcup_{p \in \mathcal{M}} \bigcup_{\psi \in [\phi]} \{(p, \psi(p))\} \\ &= \bigcup_{\psi \in [\phi]} \bigcup_{p \in \mathcal{M}} \{(p, \psi(p))\} \\ &\subseteq \mathcal{M} \times \mathbb{R} \end{aligned}$$

Projections

- Consider the following projection:

Projections $E \to \mathcal{M}$

$$\pi : \begin{cases} E & \to \mathcal{M} \\ (p, \underbrace{\psi(p)}_{\in[\phi]}) & \mapsto p \end{cases}$$

- So far, we have been trying to build bundle-related notions algebraically rather than topologically. In this light, a projection $\pi : E \to \mathcal{M}$ can be viewed as an idempotent map from E to its subset $\mathcal{M}$,

$$\pi \circ \pi = \pi$$

'Baby' Bundles

- A bundle formalizes the notion of a space living on another space (or a space parameterized by another space).
- Informally, we may imagine a bundle captures the idea of the graphs $\bigcup_{p\in\mathcal{M}} \{(p, \psi(p))\}$ of multiple fields ψ in the same gauge $[\phi]$, living on a spacetime $\mathcal{M}$.
- Such a structure (which we will call a 'baby' bundle as it does not yet incorporate topology :) is the tuple $(E, \pi, \mathcal{M})$, often simply denoted as $E \xrightarrow{\pi} \mathcal{M}$.

Visualizing Bundles

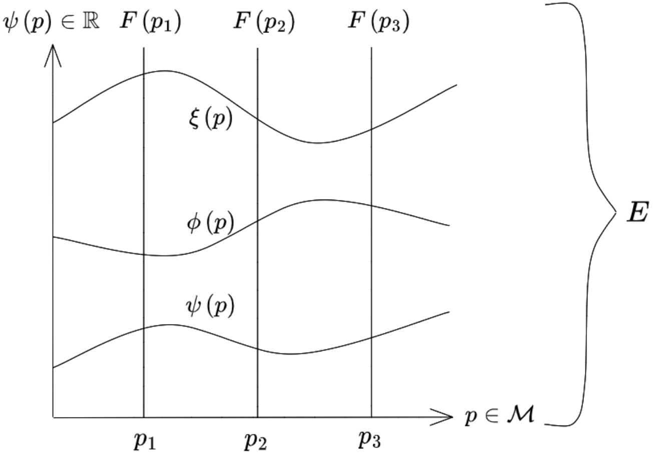

Figure: A bundle $E \xrightarrow{\pi} \mathcal{M}$. Note that $\psi \sim \phi \sim \xi$.

Topological Bundles

- In topology, a bundle is constructed by considering a total [topological] space $(E, \mathcal{O}_E)$, a base space $(B, \mathcal{O}_B)$ and a continuous surjection $\pi : E \to B$.
  (E, π, B) or $E \xrightarrow{\pi} B$ is then said to be a [topological] bundle.
- The fibre at a point $p \in B$ is defined as,

$$\begin{aligned} F(p) &:= \mathsf{preim}_{\pi}(\{p\}) \\ &:= \{x \in B : \pi(x) = \{p\}\} \end{aligned}$$

- A fibre bundle (E, B, π, F) or $F \to E \xrightarrow{\pi} B$ is a structure where $E \xrightarrow{\pi} B$ is a bundle and every fibre is homeomorphic to a manifold F, called the typical fibre of the fibre bundle,

$$\forall x \in E : \mathsf{preim}_{\pi}(\{x\}) \cong_{\mathsf{top}} F$$

Total Space

- In the field-theoretic situation we considered earlier, the total space associated with a rank (p, q) field on a spacetime $\mathcal{M}$ is typically homeomorphic to a manifold of dimension $\dim(\mathcal{M}) + p + q$.
- We will consider Newtonian gravitation and classical electrodynamics on 3-dimensional Euclidean, and 4-dimensional Minkowski space, respectively.
- In the case of the Newtonian gravitational field ϕ, the total space is $\mathbb{R}^3 \times \mathbb{R}$ and this can be equipped with the Euclidean topology $\mathcal{O}_{\mathbb{R}^4}$.
- For the electromagnetic 4-potential A, the total space is $\mathbb{R}^4 \times \mathbb{R}^4$. This is Lorentzian, but we can make it Euclidean after a Wick rotation. In other words, the total space is isomorphic to $\mathbb{R}^8$, which can then be equipped with the Euclidean topology $\mathcal{O}_{\mathbb{R}^8}$.

Product Bundle Structure

- With the above constructions, we find that the canonical projection $E \to \mathcal{M}$ we defined earlier is indeed continuous and surjective, for both the gravitational potential and electromagnetic 4-potential fields.
- Therefore, $\left(\mathbb{R}^3 \times \mathbb{R}, \pi_{\mathbb{R}^3}, \mathbb{R}\right)$ is a bundle, known as a product bundle. The same goes for $\left(\mathbb{R}^4 \times \mathbb{R}^4, \pi_{\mathbb{R}^4}, \mathbb{R}^4\right)$ in the case of the electromagnetic field in flat spacetime.
- Furthermore, in each case, the fibres are isomorphic to $\mathbb{R}$ and $\mathbb{R}^4$, respectively. This means that the product bundles above are also fibre bundles.

Sections

Definition (Section)

A [cross-]section s of a bundle $E \xrightarrow{\pi} B$ is as a continuous inverse of π,

$$\pi \circ s = \mathrm{id}_B$$

Sections can be visualized in the following manner:

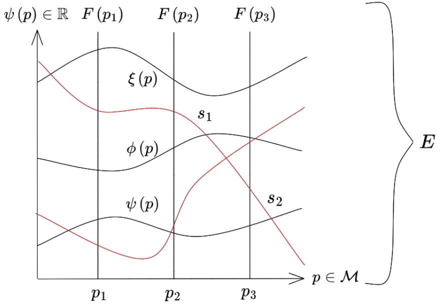

Figure: s_1 and s_2 are sections of the bundle $E \xrightarrow{\pi} B$.

- In the modern, geometric construction of classical field theory, classical fields are defined as sections of fibre bundles.
- The typical fibres of these fibre bundles are usually Lie groups (which are manifolds, as required).

References

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3. Wikipedia